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AN UPPER BOUND FOR THE NUMBER OF EULERIAN ORIENTATIONS OF A REGULAR GRAPH

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We establish a new upper bound for the number of Eulerian orientations of a regular graph with even degrees.

1. Introduction

An *Eulerian orientation* of an undirected graph G is an orientation of the edges such that the indegree of each vertex is equal to its outdegree. We denote by $\varepsilon(G)$ the number of Eulerian orientations of G. Lower and upper bounds for $\varepsilon(G)$ have been independently established in [1, 3]:

A. Schrijver has shown in [3] that

$$\varepsilon(G) \geqq \left(2^{-d}\binom{2d}{d}\right)^n \tag{S1}$$

where G is a loopless $2d$-regular graph with n vertices. The author has shown in [1][1] that

$$\varepsilon(G) \geqq 2^d\left(\frac{(2d-1)!!}{d!}\right)^{n-1} \tag{LV1}$$

The bound (LV1) is slightly better than (S1). In fact for given d, accordingly with Theorem 2 of [3] the ground numbers are equal, namely $\binom{2d}{d}$, (and cannot be improved as function of d only). The ratio is asymptotically equivalent to $\sqrt{\pi d}$.

The upper bound in [3] is

$$\varepsilon(G) \leqq \binom{2d}{d}^{n/2} \tag{S2}$$

An upper bound for $\varepsilon(G)$ is given in [1] in the case $d=2$ as an application of Proposition 4.1 (see below Lemma 2). For a connected, 4-regular, loopless graph

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[1] Errata. Proposition 5.3 in [1] is misstated: the exponent n should be $n-1$. Page 408 line 3 read $2/(d!)^n$ instead of $1/2(d!)^n$.

G of $n \geqq 4$ vertices we have

(LV2) $$\varepsilon(G) \leqq 9 \cdot 2^{n-3}$$

The ground number $\binom{2d}{d}^{0.5}$ in (S2) cannot be improved for general d-regular graphs [3] (the worst case is given by graphs whose connected components are d parallel edges). The bound (LV2), which holds for connected graphs, is strictly better $\left(\frac{9}{8} \cdot 2^n \text{ as compared to } (\sqrt{6})^n\right)$.

Our purpose in the present note is to derive a new upper bound for $\varepsilon(G)$ in terms of d, n and the maximum number of edge-disjoint circuits of G. Applied to loopless connected graphs this bound is significantly sharper than (S2).

2. Main theorem

Graphs considered in the sequel may contain loops. In evaluating $\varepsilon(G)$ we make the convention that there are two ways to direct a loop.

Theorem 1. *Let G be a connected $2d$-regular graph, $d \geqq 2$. We have*

(LV2a) $$\varepsilon(G) \leqq 2^k \left(\frac{(2d-1)!!}{d!}\right)^{n - \frac{k-1}{d-1}}$$

where n is the number of vertices of G and k is any number at least equal to the maximal number of pairwise edge-disjoint cycles of G.

The proof uses an inductive construction of [1]. Let x be a vertex of G and l be the number of loops attached to x. Let T be the set of partitions into pairs of the $2(d-l)$ non-loop edges incident to x. We associate with each $t \in T$ a graph G_t obtained from G by deleting x and all incident edges, and introducing for each pair of edges in a same set of t a new edge joining the endvertices different from x of these two edges (see [1] for more details). Clearly G_t is a $2d$-regular graph with $n-1$ vertices, and $c(G_t) \leqq c(G) - l$, where $c(G)$ denotes the maximal number of pairwise edge-disjoint cycles of G. Note that $|T| = (2d-2l-1)!! = (2d-2l-1)(2d-2l-3) \dots 5.3.1$.

Lemma 2. *We have*

$$\varepsilon(G) = \frac{2^l}{(d-l)!} \sum_{t \in T} \varepsilon(G_t).$$

Proof. By counting in two ways the elements of the set $\{(\mathcal{O}, t) : \mathcal{O}$ is an Eulerian orientation of G such that for each $t \in T$ exactly one of its edges is directed toward $x\}$ we find $2^l \sum_{t \in T} \varepsilon(G_t) = (d-l)! \varepsilon(G)$. ∎

Set

$$a(n, k) = 2^k \left(\frac{(2d-1)!!}{d!}\right)^{n - \frac{k-1}{d-1}}.$$

Lemma 3. *For given n, $a(n, k)$ is an increasing function of k.*

Proof. We have

$$a(n,k)=\left[2\left(\frac{(2d-1)!!}{d!}\right)^{-\frac{1}{d-1}}\right]^k\left(\frac{(2d-1)!!}{d!}\right)^{n+\frac{1}{d-1}}.$$

It suffices to show that

$$2\left(\frac{(2d-1)!!}{d!}\right)^{-\frac{1}{d-1}}>1.$$

Raising to the $(d-1)$-th power this inequality amounts to

$$\frac{(2d-1)!!}{d!}=\frac{2d-1}{d}\cdot\frac{2d-3}{d-1}\cdots\frac{3}{2}<2^{d-1}\quad\blacksquare$$

Proof of Theorem 1. In view of Lemma 3 it suffices to establish (LV2a) when $k=c(G)$. The proof is by induction on n.

If $n=1$ the graph G consists of d loops attached to one vertex. We have $\varepsilon(G)=2^d$. Theorem 1 with $k=c(G)=d$ is $\varepsilon(G)\leqq 2^d$. Suppose $n\geqq 2$. Let x be a vertex of G, not a cutvertex. Let l be the number of loops attached to x, $0\leqq l\leqq d-1$. By Lemma 2 we have

$$\varepsilon(G)=\frac{2^l}{(d-l)!}\sum_{t\in T}\varepsilon(G_t).$$

There are $|T|=(2d-2l-1)!!$ graphs G_t. Each G_t is connected, $2d$-regular with $n-1$ vertices and $c(G_t)\leqq c(G)-l$. Hence, using the induction hypothesis and Lemma 3 we get

$$\varepsilon(G)\leqq\frac{2^l}{(d-l)!}(2d-2l-1)!!\,a(n-1,k-l)=$$

$$=\frac{(2d-2l-1)!!}{(d-l)!}\cdot\left(\frac{(2d-1)!!}{d!}\right)^{-1+\frac{1}{d-1}}\cdot a(n,k).$$

To establish Theorem 1 it suffices to show that for $d\geqq 2$ and $0\leqq l\leqq d-1$ we have

$$\frac{(2d-2l-1)!!}{(d-l)!}\cdot\left(\frac{(2d-1)!!}{d!}\right)^{-1+\frac{1}{d-1}}\leqq 1.$$

For $l=d-1$ this inequality holds trivially. Suppose $0\leqq l\leqq d-2$. The inequality to prove can be rewritten

$$\left(\frac{(2d-2l-1)!!}{(d-l)!}\right)^{\frac{1}{d-l-1}}\leqq\left(\frac{(2d-1)!!}{d!}\right)^{\frac{1}{d-1}}.$$

Set

$$b(d)=\left(\frac{(2d-1)!!}{d!}\right)^{\frac{1}{d-1}}.$$

We show that $b(d)$ is an increasing function of d for $d \geqq 2$, integer. The inequality $b(d) < b(d+1)$ is

$$\left(\frac{(2d-1)!!}{d!}\right)^{\frac{1}{d-1}} < \left(\frac{(2d+1)!!}{(d+1)!}\right)^{\frac{1}{d}}.$$

Raising to the $d(d-1)$-th power, we get equivalently

$$\left(\frac{(2d-1)!!}{d!}\right)^{d} < \left(\frac{(2d+1)!!}{(d+1)!}\right)^{d-1}$$

or, after simplification

$$\frac{(2d-1)!!}{d!} = \frac{2d-1}{d} \cdot \frac{2d-3}{d-1} \cdots \frac{3}{2} < \left(\frac{(2d+1)!!}{(d+1)!}\right)^{d-1}. \blacksquare$$

3. Application

Theorem 4. *Let G be a connected $2d$-regular graph on n vertices with $d \geqq 2$ and girth $\geqq g$. Then*

$$\text{(LV2b)} \qquad \varepsilon(G) \leqq \left(\frac{(2d-1)!!}{d!}\right)^{\frac{1}{d-1}} \left[2^{\frac{d}{g}} \left(\frac{(2d-1)!!}{d!}\right)^{1-\frac{1}{g}\cdot\frac{d}{d-1}}\right]^{n}.$$

Proof. Since every cycle of G is of length $\geqq g$, we have $c(G) \leqq \frac{dn}{g}$. Applying Theorem 1 with $k = \frac{dn}{g}$, we get (LV2b). $\blacksquare$

When $d \to \infty$ the ratio of the ground numbers in (S2) and (LV2b) has the asymptotic value

$$2^{\frac{1}{g}} (\sqrt{\pi d})^{\frac{1}{2}-\frac{1}{g}}.$$

A loopless connected graph has girth $g \geqq 2$. In this case by Theorem 4 we have

$$\text{(LV2b)} \qquad \varepsilon(G) \leqq \left(\frac{(2d-1)!!}{d!}\right)^{\frac{1}{d-1}} \left[2^{\frac{d}{2}} \left(\frac{(2d-1)!!}{d!}\right)^{\frac{1}{2}\cdot\frac{d-2}{d-1}}\right]^{n}.$$

As

$$2^{d}\left(\frac{(2d-1)!!}{d!}\right) = \binom{2d}{d},$$

the ground number in (LV2b) is strictly smaller than the ground number in (S2). The limit of the ratio when $d \to \infty$ is $\sqrt{2}$. In particular (LV2b) implies (S2) for connected graphs. For $d=2$ (LV2b) is $\varepsilon(G) \leqq \frac{3}{2} \cdot 2^{n}$, slightly weaker than (LV2) (obtained by a finer use of Lemma 2).

Example. Applying Theorem 4 with $g=3$ and $n=2d+1$ to the complete graph K_n on n vertices, we get an upper bound (LV2b) of the asymptotic form

$$4^{d^2}d^{-\frac{2d}{3}(1+o(d))} \quad \text{when} \quad d\to\infty.$$

In this case (S2) has the form $4^{d^2}d^{-\frac{d}{2}(1+o(d))}$, (S1) and (LV1) have the form $4^{d^2}d^{-d(1+o(d))}$.

The exact asymptotic value of $\varepsilon(K_n)$ has recently been obtained by B. McKay [2]. We have

$$\varepsilon(K_n)\cong\left(\frac{2^{n+1}}{\pi n}\right)^{\frac{n-1}{2}} n^{\frac{1}{2}} e^{-\frac{1}{2}}$$

which is of the form $4^{d^2}d^{-d(1+o(d))}$.

References

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